

UCLA/00/TEP/14
January 2000

$SL_q(3)$ Fields

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Abstract. The q -field theories are constructed by substituting quantum groups for the usual Lie groups. In earlier papers this construction was carried out for the quantum group $SU_q(2)$. Here the investigation is extended to $SL_q(3)$. The resulting theory describes two sectors, one sector lying close to the standard theory and accessible by perturbation theory, while the second sector describes particles that should be difficult to detect and become invisible in the $q = 1$ limit. In this note we discuss these hypothetical particles: three quark-like spinor particles coupled to three gluon-like vector particles.

PACS numbers 81R50, 81T13.

1. Since the Lie groups may be considered as degenerate forms of the quantum groups, it may be of interest to generalize the symmetry of a conventional field theory by replacing its Lie group by the corresponding quantum group. When this generalization is carried out, it is found that the state space must be expanded to describe additional degrees of freedom that are absent in the corresponding Lie theory.

Our earlier work was based on the simple $SL_q(2)$ quantum group.¹ Since the phenomenology of elementary particles requires Lie groups of higher rank, we shall here extend the earlier discussion of $SL_q(2)$ to describe fields that lie in the $SL_q(3)$ algebra.

2. In general the quantum groups may be defined by the relations²

$$RT_1T_2 = T_2T_1R \quad (2.1)$$

where

$$T_1 = T \otimes I \quad (2.2)$$

$$T_2 = I \otimes T \quad (2.3)$$

and the R matrix satisfies the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} . \quad (2.4)$$

Written out, (2.1) reads

$$\sum_{m,p=1}^n R_{ij,mp} t_{mk} t_{pl} = \sum_{m,p=1}^n t_{jp} t_{im} R_{mp,kl} \quad (2.5)$$

where for the series A_{n-1}

$$R = q \sum_{i=1}^n e_{ii} \otimes e_{ii} + \sum_{\substack{i,j=1 \\ i \neq j}}^n e_{ii} \otimes e_{jj} + (q - q_1) \sum_{\substack{i,j=1 \\ i > j}}^n e_{ij} \otimes e_{ji} . \quad (2.6)$$

Here

$$[e_{mn}]_{ij} = \delta_{mi} \delta_{nj} \quad \text{and} \quad q_1 = q^{-1} . \quad (2.7)$$

For this choice of R Eq. (2.5) becomes

$$\begin{aligned} & q t_{ik} t_{j\ell} \delta_{ij} + t_{ik} t_{j\ell} (1 - \delta_{ij}) + (q - q_1) t_{jk} t_{i\ell} \theta(i - j) \\ & = q t_{jk} t_{i\ell} \delta_{k\ell} + t_{j\ell} t_{ik} (1 - \delta_{k\ell}) + (q - q_1) t_{jk} t_{i\ell} \theta(\ell - k) \end{aligned}$$

where

$$\begin{aligned} \theta(s) &= 1 \quad s > 0 \\ &= 0 \quad s \leq 0 . \end{aligned}$$

By (2.8) one finds the following special cases, depending on whether the two matrix elements are in the same row ($i = j$):

$$\ell < k \quad t_{i\ell}t_{ik} = qt_{ik}t_{i\ell} \quad (2.9)$$

the same column ($k = \ell$):

$$j < i \quad t_{ik}t_{jk} = qt_{jk}t_{ik} \quad (2.10)$$

neither same row nor same column

$$i < j, k < \ell \quad t_{ik}t_{j\ell} = t_{j\ell}t_{ik} + (q - q_1)t_{jk}t_{i\ell} \quad (2.11)$$

$$i < j, k > \ell \quad t_{ik}t_{j\ell} = t_{j\ell}t_{ik} \quad (2.12)$$

These results may be summarized as follows: Choose any rectangle in the full $N \times N$ matrix, say

$$\begin{pmatrix} ik & i\ell \\ jk & j\ell \end{pmatrix} \quad \text{in} \quad \begin{pmatrix} t_{11} & \dots & t_{1N} \\ t_{N1} & \dots & t_{NN} \end{pmatrix} \quad (2.13)$$

Then ($i < j$ and $k < \ell$)

$$t_{ik}t_{i\ell} = qt_{i\ell}t_{ik} \quad (2.14)$$

$$t_{ik}t_{jk} = qt_{jk}t_{ik} \quad (2.15)$$

$$(t_{ik}, t_{j\ell}) = (q - q_1)t_{jk}t_{i\ell} \quad (2.16)$$

$$(t_{i\ell}, t_{jk}) = 0 \quad (2.17)$$

i.e. the 4 vertices exhibited in (2.13) belong to the algebra of $GL_q(2)$. Note that all the commuting elements lie on lines of positive slope. Therefore the maximum set of commuting elements lie on the minor diagonal.

There is also the quantum determinant

$$\det_q T = \sum_{\sigma} (-q)^{\ell(\sigma)} t_{1\sigma_1} \dots t_{n\sigma_n} \quad \sigma \in \text{symm}(n) \quad (2.18)$$

where $\ell(\sigma)$ is the number of inversions in the permutation σ .

The quantum determinant commutes with all elements of T

$$(\det_q T, t_{ij}) = 0. \quad (2.19)$$

3. The quantum group $GL_q(3)$.

Let us introduce the following notation

$$T = \begin{pmatrix} E_+(1, 1) & E_+(1, 2) & H(1) \\ E_+(2, 1) & H(2) & E_-(2, 3) \\ H(3) & E_-(3, 2) & E_-(3, 3) \end{pmatrix} \quad (3.1)$$

Consider a matrix realization of the elements of T where the commuting set $\{H\}$ is Hermitian:

$$H_i = \bar{H}_i . \quad (3.2)$$

By (2.14) and (2.15)

$$E_+ H = q H E_+ \quad (3.3)$$

$$E_- H = q_1 H E_- \quad (3.4)$$

where E and H share either a row or column index. By Hermitian conjugation of (3.3) and (3.4)

$$H \bar{E}_+ = \bar{q} \bar{E}_+ H \quad (3.5)$$

$$H \bar{E}_- = \bar{q}_1 \bar{E}_- H \quad (3.6)$$

Therefore we may set

$$q = \bar{q} \quad \text{and} \quad \bar{E}_+ = E_- . \quad (3.7)$$

Then

$$T = \begin{pmatrix} \bar{E}(1) & \bar{E}(2) & H(1) \\ \bar{E}(3) & H(2) & E(2) \\ H(3) & E(3) & E(1) \end{pmatrix} . \quad (3.8)$$

Note that $E(1)$ and $\bar{E}(1)$ may be interchanged if q is replaced by $1/q$. We also have

$$\begin{aligned} (E(2), E(3)) &= 0 \\ (H(1), E(3)) &= 0 \\ (H(3), E(2)) &= 0 \\ (H(i), H(j)) &= 0 \quad i, j = 1, 2, 3 \end{aligned} \quad (3.9)$$

4. Representation of the algebra.

Introduce the basis states $|n_i\rangle$, eigenstates of the H_i . Then

$$E_+ H_i |n_i\rangle = q H_i E_+ |n_i\rangle \quad (4.1)$$

or

$$H_i (E_+ |n_i\rangle) = q_1 h_i(n_i) (E_+ |n_i\rangle) . \quad (4.2)$$

Define $|n_i + 1\rangle$ by

$$E_+ |n_i\rangle = \lambda(n_i) |n_i + 1\rangle . \quad (4.3)$$

Then

$$H_i |n_i + 1\rangle = q_1 h_i(n_i) |n_i + 1\rangle \quad (4.4)$$

or the eigenvalues of H_i are given by

$$h_i(n_i + 1) = q_1 h_i(n_i) \quad (4.5)$$

and

$$h_i(n_i) = q_1^{n_i} h_i(0) \quad (4.6)$$

Likewise

$$\begin{aligned} H(E_-|n\rangle) &= qh(n)(E_-|n\rangle) \\ &= h(n-1)(E_-|n\rangle) . \end{aligned} \quad (4.7)$$

Therefore set

$$E_-|n_i\rangle = \mu(n_i)|n_i - 1\rangle \quad (4.8)$$

$$E_-|0\rangle = 0 . \quad (4.9)$$

In the above equations if

$$\begin{aligned} E_+ \text{ is } \bar{E}(1) , \text{ then } H \text{ is } H(1) \text{ or } H(3) \\ E_+ \text{ is } \bar{E}(2) , \text{ then } H \text{ is } H(1) \text{ or } H(2) \\ E_+ \text{ is } \bar{E}(3) , \text{ then } H \text{ is } H(2) \text{ or } H(3) \end{aligned} \quad (4.10)$$

The complete notation for the basis states is

$$|\vec{n}\rangle = |n_1, n_2, n_3\rangle \quad (4.11)$$

where (n_1, n_2, n_3) refer to (H_1, H_2, H_3) . In this notation (4.8) reads

$$E_-(2)|n_1, n_2, n_3\rangle = \mu_2(n_1, n_2, n_3)|n_1 - 1, n_2 - 1, n_3\rangle \quad (4.12)$$

$$E_-(3)|n_1, n_2, n_3\rangle = \mu_3(n_1, n_2, n_3)|n_1, n_2 - 1, n_3 - 1\rangle \quad (4.13)$$

$$E_-(1)|n_1, n_2, n_3\rangle = \mu_1(n_1, n_2, n_3)|n_1 - 1, n_2 - 2, n_3 - 1\rangle \quad (4.14)$$

$$E_-(2)|0, 0, n_3\rangle = 0 \quad E_-(1)|0, n_2, 0\rangle = 0 \quad (4.15)$$

$$E_-(3)|n_1, 0, 0\rangle = 0 \quad (4.15)$$

In this representation $E(1)$ behaves differently from $E(2)$ and $E(3)$ since

$$(\bar{E}(1), H(2)) = \tilde{q} \bar{E}(2)\bar{E}(3) , \quad \tilde{q} = q - q_1 . \quad (4.16)$$

There is no corresponding equation for $\bar{E}(2)$ or $\bar{E}(3)$. Then (4.16) implies

$$\begin{aligned} [h(n_2) - h(n'_2)]\langle\vec{n}'|\bar{E}(1)|\vec{n}\rangle &= \tilde{q}\langle\vec{n}'|\bar{E}(2)\bar{E}(3)|\vec{n}\rangle \\ &= \tilde{q}\langle n_1 + 1, n_2 + 2, n_3 + 1|\bar{E}(2)|n_1, n_2 + 1, n_3 + 1\rangle\langle n_1, n_2 + 1, n_3 + 1|\bar{E}(3)|n_1, n_2, n_3\rangle . \end{aligned} \quad (4.17)$$

where $\vec{n}' = (n_1 + 1, n_2 + 2, n_3 + 1)$. Of the three E -operators only $E(1)$ moves all three quantum numbers. The raising and lowering operators (determining transition amplitudes) are subject to simple selection rules that may be found as follows:

For the E and H shown in (3.8)

$$\langle n|\bar{E}H|m\rangle = q\langle n|H\bar{E}|m\rangle \quad (4.18)$$

$$\langle n|\bar{E}|m\rangle[h(m) - qh(n)] = 0 \quad (4.19)$$

$$\langle n|\bar{E}|m\rangle[q_1^m - q_1^{n-1}] = 0 \quad (4.20)$$

$$\langle n|\bar{E}|m\rangle = 0, \quad m \neq n-1 \quad (4.21)$$

Similarly

$$\langle n|E|m\rangle = 0 \quad m \neq n+1. \quad (4.22)$$

These selection rules are shown in (4.12) and (4.13) for $E(2)$ and $E(3)$. There is an additional selection rule shown in (4.14) and (4.17) for $E(1)$.

5. Restrictions on the Transition Amplitudes and the Value of q .

The transition amplitudes are restricted by the following relations

$$(\bar{E}(1), E(1)) = \tilde{q} H(1)H(3) \quad \tilde{q} = q - q_1 \quad (5.1)$$

$$(\bar{E}(2), E(2)) = \tilde{q} H(1)H(2) \quad (5.2)$$

$$(\bar{E}(3), E(3)) = \tilde{q} H(2)H(3) \quad (5.3)$$

Since these relations have a common structure we first discuss only (5.1). The diagonal element of this equation may be written as follows:

$$\langle n_1 n_2 n_3 | (\bar{E}(1), E(1)) | n_1 n_2 n_3 \rangle = \tilde{q} \langle n_1 n_2 n_3 | H(1)H(3) | n_1 n_2 n_3 \rangle \quad (5.4)$$

where $|n_1 n_2 n_3\rangle$ is a common eigenstate of $H(1)$, $H(2)$, and $H(3)$.

Eq. (5.4) becomes

$$\sum_{\vec{p}} |\langle \vec{p} | E(1) | \vec{n} \rangle|^2 - \sum_{\vec{p}} |\langle \vec{n} | E(1) | \vec{p} \rangle|^2 = \tilde{q} h^{(n_1)}(1) h^{(n_3)}(3). \quad (5.5)$$

If $\vec{n} = (n_1, n_2, n_3)$, then by (4.14)

$$\begin{aligned} \langle \vec{p} | E(1) | \vec{n} \rangle &= 0 \quad \text{unless} \quad \vec{p} = (n_1 - 1, n_2 - 2, n_3 - 1) \\ \langle \vec{n} | E(1) | \vec{p} \rangle &= 0 \quad \text{unless} \quad \vec{p} = (n_1 + 1, n_2 + 2, n_3 + 1). \end{aligned} \quad (5.6)$$

Hence, denoting the lowest eigenvalue of H_i by α_i we have

$$\begin{aligned} &|\langle n_1 - 1, n_2 - 2, n_3 - 1 | E(1) | n_1, n_2, n_3 \rangle|^2 - |\langle n_1, n_2, n_3 | E(1) | n_1 + 1, n_2 + 2, n_3 + 1 \rangle|^2 \\ &= \tilde{q} \alpha_1 \alpha_3 q_1^{n_1} q_1^{n_3} \end{aligned} \quad (5.7)$$

or

$$f(n_1 - 1, n_2 - 2, n_3 - 1) - f(n_1, n_2, n_3) = g(n_1, n_3) \quad (5.8)$$

where

$$f(n_1, n_2, n_3) = |\langle n_1, n_2, n_3 | E(1) | n_1 + 1, n_2 + 2, n_3 + 1 \rangle|^2 \quad (5.9)$$

$$g(n_1, n_3) = \tilde{q} \alpha_1 \alpha_3 q^{-n_1 - n_3} \quad (5.10)$$

Set $n_1 = n_2 = n_3 = 0$ in (5.8). Then

$$f(-1, -2, -1) - f(0, 0, 0) = g(0, 0) .$$

But $f(-1, -2, -1) = 0$ since $|000\rangle$ is the vacuum state. Then

$$f(0, 0, 0) = -g(0, 0) = -\tilde{q} \alpha_1 \alpha_3 . \quad (5.11)$$

By (5.8)

$$f(n_1 + 1, n_2 + 2, n_3 + 1) = f(n_1, n_2, n_3) - g(n_1 + 1, n_3 + 1)$$

and

$$f(n_1 + m, n_2 + 2m, n_3 + m) = f(n_1, n_2, n_3) - \sum_{s=1}^m g(n_1 + s, n_3 + s) . \quad (5.12)$$

Set $n_1 = n_2 = n_3 = 0$. Then

$$f(m, 2m, m) = f(0, 0, 0) - \sum_{s=1}^m g(s, s) \quad (5.13)$$

$$= -\tilde{q} \alpha_1 \alpha_3 \langle m \rangle_{q^{-2}} \quad (5.14)$$

by (5.10) where

$$\langle m \rangle_{q^{-2}} = \frac{q^{-2m} - 1}{q^{-2} - 1} . \quad (5.15)$$

By (5.9) and (5.14)

$$|\langle n, 2n, n | E(1) | n + 1, 2(n + 1), n + 1 \rangle|^2 = -\tilde{q} \alpha_1 \alpha_3 \langle n \rangle_{q^{-2}} \quad (5.16)$$

The corresponding discussion for $E(2)$ and $E(3)$ is simpler since only two quantum numbers change when $E(2)$ and $E(3)$ act, while all three $(n_1 \ n_2 \ n_3)$ are changed when $E(1)$ acts. The results for $E(2)$ and $E(3)$ are as follows:

$$|\langle n, n, n | E(2) | n + 1, n + 1, n \rangle|^2 = -\tilde{q} \alpha_2 \alpha_1 \langle n \rangle_{q^{-2}} \quad (5.17)$$

$$|\langle n, n, n | E(3) | n, n + 1, n + 1 \rangle|^2 = -\tilde{q} \alpha_2 \alpha_3 \langle n \rangle_{q^{-2}} . \quad (5.18)$$

Since the left sides of (5.16)-(5.18) are positive one has

$$-\tilde{q} \alpha_i \alpha_j > 0 \quad i, j = 1, 2, 3 \quad i \neq j \quad (5.19)$$

The preceding equations (5.19) imply that $\alpha_i \alpha_j$ always has the same sign. Therefore the α_i are also all of the same sign, the $\alpha_i \alpha_j$ are positive, and

$$-\tilde{q} > 0$$

or by (5.1)

$$q < 1 . \quad (5.20)$$

6. The Quantum Determinant.

The magnitudes of the transition amplitudes may be restricted by the quantum determinant. The quantum determinant corresponding to (3.8) and computed according to (2.18) is

$$\begin{aligned}\Delta = & \bar{E}(1)H(2)E(1) - q[\bar{E}(2)\bar{E}(3)E(1) + \bar{E}(1)E(2)E(3)] \\ & - q^3H(3)H(2)H(1) \\ & + q^2[\bar{E}(2)E(2)H(3) + H(1)\bar{E}(3)E(3)] .\end{aligned}\tag{6.1}$$

Since Δ belongs to the center of the algebra it may be set equal to unity:

$$\Delta = 1 .\tag{6.2}$$

Applied to the H -vacuum we have

$$\Delta|0\rangle = |0\rangle\tag{6.3}$$

implying

$$-q^3h_o(3)h_o(2)h_o(1) = 1 .\tag{6.4}$$

Let us again set

$$h_o(i) = \alpha_i .\tag{6.5}$$

Then

$$\alpha_1\alpha_2\alpha_3 = -q_1^3 .\tag{6.6}$$

Since α_1, α_2 and α_3 all have the same sign by (5.19), they are all negative.

In general

$$\langle \vec{n} | \Delta | \vec{p} \rangle = \langle \vec{n} | \Delta | \vec{n} \rangle \delta(\vec{n}, \vec{p}) .\tag{6.7}$$

To evaluate $\langle \vec{n} | \Delta | \vec{n} \rangle$ we need

$$\langle \vec{n} | \bar{E}(1)H(2)E(1) | \vec{n} \rangle = \alpha_2 q_1^{n_2-2} |\langle \vec{p}_1 | E(1) | \vec{n} \rangle|^2\tag{6.8}$$

and

$$\langle \vec{n} | \bar{E}(2)\bar{E}(3)E(1) + \bar{E}(1)E(2)E(3) | \vec{n} \rangle = 2\alpha_2 q_1^{n_2-1} |\langle \vec{p}_1 | E(1) | \vec{n} \rangle|^2 .\tag{6.9}$$

To obtain (6.9) we have used (6.8) and

$$\tilde{q}E(2)E(3) = (H(2), E(1)) .\tag{6.10}$$

Then

$$\begin{aligned}\langle \vec{n} | \Delta | \vec{n} \rangle = & -\alpha_2 q_1^{n_2-2} |\langle \vec{p}_1 | E(1) | \vec{n} \rangle|^2 - q^3 \alpha_1 \alpha_2 \alpha_3 q^{n_1+n_2+n_3} \\ & + q^2 [\alpha_3 q_1^{n_3} |\langle \vec{p}_2 | E(2) | \vec{n} \rangle|^2 + \alpha_1 q_1^{n_1} |\langle \vec{p}_3 | E(3) | \vec{n} \rangle|^2]\end{aligned}\tag{6.11}$$

or by (6.3)

$$\begin{aligned}1 - q_1^{n_1+n_2+n_3} = & q^2 \{ \alpha_1 q_1^{n_1} |\langle \vec{p}_3 | E(3) | \vec{n} \rangle|^2 + \alpha_3 q_1^{n_3} |\langle \vec{p}_2 | E(2) | \vec{n} \rangle|^2 \\ & - \alpha_2 q_1^{n_2} |\langle \vec{p}_1 | E(1) | \vec{n} \rangle|^2 \}\end{aligned}\tag{6.12}$$

where

$$\begin{aligned}\vec{p}_1 &= (n_1 - 1, n_2 - 2, n_3 - 1) \\ \vec{p}_2 &= (n_1 - 1, n_2 - 1, n_3) \\ \vec{p}_3 &= (n_1, n_2 - 1, n_3 - 1)\end{aligned}\tag{6.13}$$

Note that the curly bracket in (6.12) is negative since $q_1 > 1$.

If $\vec{n} = (n, n, n)$ then by (6.12) and (5.16)-(5.18)

$$1 - q_1^{3n} = q^2(-\tilde{q}\alpha_1\alpha_2\alpha_3)q_1^n \langle n \rangle_{q_1^2}\tag{6.14}$$

or

$$q_1^n = 1\tag{6.15}$$

by (6.6).

Since q_1 is real, $n = 0$. It follows that there are no solutions of (6.2) for any state $|nnn\rangle$ except the vacuum state $|000\rangle$. Therefore the restriction on the q -determinant acts as an exclusion principle forbidding any state $|nnn\rangle$ except $|000\rangle$ in which the quantum numbers of the three H -particles agree.

7. Invariant Forms.

Let us restate (2.5) to make explicit the distinction between T_1 and T_2

$$\sum_{mp} R_{ijmp} T(1)_{mk} T(2)_{p\ell} = \sum_{mp} T(2)_{jp} T(1)_{im} R_{mpk\ell} .\tag{7.1}$$

Then

$$\begin{aligned}\sum_{\substack{mp \\ k\ell}} R_{ijmp} T(1)_{mk} T(2)_{p\ell} (T(2)^{-1})_{\ell s} (T(1)^{-1})_{kt} \\ = \sum_{\substack{mp \\ k\ell}} T(2)_{jp} T(1)_{im} R_{mpk\ell} (T(2)^{-1})_{\ell s} (T(1)^{-1})_{kt}\end{aligned}\tag{7.2}$$

or

$$R_{ijts} = \sum_{\substack{mp \\ k\ell}} T(2)_{jp} T(1)_{im} R_{mpk\ell} (T(2)^{-1})_{\ell s} (T(1)^{-1})_{kt} .\tag{7.3}$$

Consider two fields $\psi(x)$ and $\chi(x)$ and the following associated bilinear

$$\begin{aligned}\sum \psi_{ij}(x) R_{ijts} \chi_{ts}(x) &= \sum (\psi_{ij}(x) T(2)_{jp} T(1)_{im} R_{mpk\ell} (T(2)^{-1})_{\ell s} (T(1)^{-1})_{kt} \chi_{ts}(x)) \\ &= \sum \psi'_{mp}(x) R_{mpk\ell} \chi'_{k\ell}(x)\end{aligned}\tag{7.4}$$

by (7.3), where

$$\psi'_{mp} = \sum \psi_{ij} T(2)_{jp} T(1)_{im}\tag{7.5}$$

$$\chi'_{k\ell} = \sum (T(2)^{-1})_{\ell s} (T(1)^{-1})_{kt} \chi_{ts}\tag{7.6}$$

Then $\psi R\chi$ is a bilinear invariant under the T -transformations (7.5) and (7.6).

There is also an invariant trilinear interaction stemming from the invariance of the quantum determinant. To show this, rewrite (2.18) as follows:

$$\Delta^q = \sum_{j_1 \dots j_n} (-q)^{-\ell(i_1 \dots i_n)} t_{i_1 j_1} \dots t_{i_n j_n} (-q)^{\ell(j_1 \dots j_n)} \quad (7.7)$$

where we count the number of inversions between $(i_1 \dots i_n)$ and $(j_1 \dots j_n)$ rather than the number of inversions between $(1 \dots n)$ and $(\sigma_1 \dots \sigma_n)$ as in (2.18). Then

$$\sum_{j_1 \dots j_n} (-q)^{\ell(j)} t_{i_1 j_1} \dots t_{i_n j_n} = (-q)^{\ell(i)} \Delta^q. \quad (7.8)$$

When written to resemble the familiar formula for the usual determinant, (7.8) becomes

$$\sum_{j_1 \dots j_n} \mathcal{E}_{j_1 \dots j_n}^q t_{i_1 j_1} \dots t_{i_n j_n} = \mathcal{E}_{i_1 \dots i_n}^q \Delta^q \quad (7.9)$$

where

$$\mathcal{E}_{j_1 \dots j_n}^q = (-q)^{\ell(j_1 \dots j_n)} \quad (7.10a)$$

and

$$\mathcal{E}_{j_1 \dots j_n}^q = 0 \quad \text{unless all indices are different.} \quad (7.10b)$$

The \mathcal{E}^q symbol thus behaves as an invariant tensor with weight Δ^q .

The trilinear form

$$I_3 = \mathcal{E}_{j_1 j_2 j_3}^q \varphi^{j_1} \varphi^{j_2} \varphi^{j_3}$$

is invariant under

$$\varphi^{j'} = \varphi^i T_i^j$$

where the φ^i commute with the T_i^j since

$$\begin{aligned} I'_3 &= \mathcal{E}_{j_1 j_2 j_3}^q (\varphi^{j_1} \varphi^{j_2} \varphi^{j_3})' \\ &= \mathcal{E}_{j_1 j_2 j_3}^q (\varphi^{i_1} \varphi^{i_2} \varphi^{i_3} T_{i_1}^{j_1} T_{i_2}^{j_2} T_{i_3}^{j_3}) \\ &= (\mathcal{E}_{j_1 j_2 j_3}^q T_{i_1}^{j_1} T_{i_2}^{j_2} T_{i_3}^{j_3}) \varphi^{i_1} \varphi^{i_2} \varphi^{i_3} \\ &= \Delta^q \mathcal{E}_{i_1 i_2 i_3}^q \varphi^{i_1} \varphi^{i_2} \varphi^{i_3} \end{aligned}$$

or

$$I'_3 = \Delta^q I_3 \quad (7.11)$$

by (7.9).

The invariance of the quantum determinant may also be used to form the inverse of T that is needed in (7.2). By (7.9) with $\Delta^q = 1$

$$\sum_{j_1} t_{i_1 j_1} \sum_{j_2 \dots j_n} \mathcal{E}_{j_1 \dots j_n}^q t_{i_2 j_2} \dots t_{i_n j_n} = \mathcal{E}_{i_1 \dots i_n}^q \quad (7.12)$$

i.e.

$$\sum_{j_1} t_{i_1 j_1} \sum_{j_2 \dots j_n} t_{i_2 j_2} \dots t_{i_n j_n} (-q)^{\ell \left(\begin{smallmatrix} 1 & \dots & n \\ j_1 & \dots & j_n \end{smallmatrix} \right)} = (-q)^{\ell \left(\begin{smallmatrix} 1 & \dots & n \\ i_1 & \dots & i_n \end{smallmatrix} \right)} . \quad (7.13)$$

Then

$$\begin{aligned} \sum_{j_1} t_{i_1 j_1} \sum_{\substack{j_2 \dots j_n \\ i_2 \dots i_n}} t_{i_2 j_2} \dots t_{i_n j_n} (-q)^{\ell \left(\begin{smallmatrix} 1 & \dots & n \\ j_1 & \dots & j_n \end{smallmatrix} \right)} (-q)^{-\ell \left(\begin{smallmatrix} 12 & \dots & n \\ si_2 & \dots & i_n \end{smallmatrix} \right)} \\ = \sum_{i_2 \dots i_n} (-q)^{\ell \left(\begin{smallmatrix} 1 & \dots & n \\ i_1 & \dots & i_n \end{smallmatrix} \right) - \ell \left(\begin{smallmatrix} 1 & \dots & n \\ si_2 & \dots & i_n \end{smallmatrix} \right)} \end{aligned} \quad (7.14)$$

or

$$\begin{aligned} \sum_{j_1} t_{i_1 j_1} \sum_{\substack{j_1 \dots j_n \\ i_2 \dots i_n}} t_{i_2 j_2} \dots t_{i_n j_n} (-q)^{\ell \left(\begin{smallmatrix} si_2 & \dots & i_n \\ j_1 j_2 & \dots & j_n \end{smallmatrix} \right)} = \sum_{i_2 \dots i_n} (-q)^{\ell \left(\begin{smallmatrix} si_2 & \dots & i_n \\ i_1 i_2 & \dots & i_n \end{smallmatrix} \right)} \\ = (n-1)! \delta_{i_1}^s \end{aligned} \quad (7.15)$$

since

$$\ell \left(\begin{smallmatrix} 1 & \dots & n \\ j_1 & \dots & j_n \end{smallmatrix} \right) = \ell \left(\begin{smallmatrix} 1 & \dots & n \\ i_1 & \dots & i_n \end{smallmatrix} \right) + \ell \left(\begin{smallmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{smallmatrix} \right) . \quad (7.16)$$

Therefore the matrix inverse to T is²

$$(T^{-1})_{j_1 s} = \frac{1}{(n-1)!} \sum_{\substack{i_2 \dots i_n \\ j_2 \dots j_n}} t_{i_2 j_2} \dots t_{i_n j_n} (-q)^{\ell \left(\begin{smallmatrix} si_2 & \dots & i_n \\ j_1 j_2 & \dots & j_n \end{smallmatrix} \right)} \quad (7.17)$$

8. Field Theory.

In an earlier paper¹ the following field theoretic Lagrangian was proposed

$$\mathcal{L} = -\frac{1}{4} \sum_{\alpha} L(\alpha) \vec{F}_{\mu\nu} \vec{F}^{\mu\nu} R(\alpha) + i \tilde{\psi} C \epsilon \gamma^{\mu} \vec{\nabla}_{\mu} \psi + \frac{1}{2} [(\tilde{\varphi} \overleftarrow{\nabla}_{\mu}) \epsilon (\vec{\nabla}_{\mu} \varphi) + \tilde{\varphi} \epsilon \varphi] \quad (8.1)$$

which is both Lorentz and $SU_q(2)$ invariant. Here

$$T^t \epsilon T = T \epsilon T^t = \epsilon \quad \det_q T = 1 .$$

$$\begin{aligned}
L(\alpha)' &= L(\alpha)T^{-1} & \varphi' &= T\varphi & (\vec{\nabla}\varphi)' &= T(\vec{\nabla}\varphi) \\
R(\alpha)' &= TR(\alpha) & \tilde{\varphi}' &= \tilde{\varphi}T^t & (\tilde{\varphi}\overleftarrow{\nabla})' &= (\varphi\overleftarrow{\nabla})T^t \\
(L\epsilon)' &= (L\epsilon)T^t & \psi' &= T\psi & (\vec{\nabla}\psi)' &= T(\vec{\nabla}\psi) \\
& & \tilde{\psi}' &= \tilde{\psi}T^t & (\tilde{\psi}\overleftarrow{\nabla})' &= (\tilde{\psi}\overleftarrow{\nabla})T^t
\end{aligned} \tag{8.2}$$

Kinetic terms in $L(\alpha)$ and $R(\alpha)$, as well as other possible terms in \overleftarrow{A} have not been expressed in (8.1).

The invariance of the Lagrangian requires distinct left and right fields since F does not commute with T . In the limit $q = 1$ of (4.2) the L and R fields may be summed out as follows:

$$\sum_{\alpha} L_i(\alpha)R_j(\alpha) = \delta_{ij}$$

where the sum is over the complete set of left and right fields. Then

$$\lim_{q \rightarrow 1} \sum_{\alpha} L_i(\alpha)(FF)_{ij}R_j(\alpha) = \text{Tr } FF . \tag{8.3}$$

To pass from (8.1) to Lagrangians with symmetries of higher rank, one may replace the bilinear invariants $\tilde{\varphi}\epsilon\varphi$ and $\tilde{\psi}C\epsilon\nabla\psi$ by $\tilde{\varphi}R\varphi$ and $\tilde{\psi}CR\nabla\psi$ where the invariance of the R -forms is described in Eqs. (7.4)-(7.6).

Now assume that the generic field lies in the algebra:

$$\psi_{k\ell}(x) = \sum \varphi_{k\ell}^{\alpha\beta} t_{\alpha\beta} \tag{8.4}$$

$$\tilde{\psi}_{mp}(x) = \sum t_{\alpha\beta} \tilde{\varphi}_{mp}^{\alpha\beta} , \tag{8.5}$$

where the φ do not lie in the algebra. In a general gauge transformation

$$\tilde{\psi}'_{mp}(x) = \sum \tilde{\psi}_{ij}(x)T(2)_{jp}T(1)_{im} \tag{8.6}$$

$$\psi'_{k\ell}(x) = \sum (T(2)^{-1})_{\ell s}(T(1)^{-1})_{kt}\psi_{ts}(x) . \tag{8.7}$$

Then

$$\sum \tilde{\psi}_{ij}(x)R_{ijts}\psi_{ts}(x) = \sum \tilde{\psi}'_{mp}(x)R_{mpk\ell}\psi'_{k\ell}(x) . \tag{8.8}$$

Distinguish between the $H(i, i)$ and the $E(i, j)$ fields by associating the $H(i, i)$ with a Lorentz spinor field and $E(i, j)$ with a Lorentz vector field in a particular gauge. In this special gauge, one may write the spinor field as

$$\psi = \sum_{i=1}^3 \varphi^{\alpha} H(\alpha) , \quad \alpha = 1, 2, 3 \tag{8.9}$$

and a mass term as

$$M\tilde{\psi}CR\psi = M \sum_{i=1}^3 H(\alpha)^2 \tag{8.10}$$

if we orthonormalize as follows:

$$\tilde{\varphi}^\alpha C R \varphi^\beta = \delta^{\alpha\beta} \quad (8.11)$$

Here $\tilde{\psi}$ is a transposed spinor and C is the charge conjugation matrix.

The eigenvalue of this term in the state $|n_1 n_2 n_3\rangle$ is by (4.6)

$$M \sum_{i=1}^3 q^{-2n_i} \quad (8.12)$$

and the masses associated with the three-spinor fields are

$$m_i = q^{-2n_i} M . \quad (8.13)$$

If the field particles are all in their ground states, then by (6.6) their masses are related by

$$m_1 m_2 m_3 = M^3 q_1^6 . \quad (8.14)$$

Each field particle may also exist in excited states according to (8.13).

These three spinor fields interact through terms like

$$\tilde{\psi} C R \nabla \psi \quad (8.15)$$

where

$$\nabla_\mu = \partial_\mu + A_\mu \quad \nabla = \gamma^\mu \nabla_\mu . \quad (8.16)$$

The q -invariance of the interaction term requires

$$\tilde{\psi}' C R \nabla' \psi' = \tilde{\psi} C R \nabla \psi . \quad (8.17)$$

Let us abbreviate (8.6) and (8.7)

$$\tilde{\psi}' = \tilde{\psi} \overleftarrow{\mathcal{T}} \quad (8.18)$$

$$\psi' = \vec{\mathcal{T}} \psi \quad (8.19)$$

We shall also assume

$$(\nabla \psi)' = \vec{\mathcal{T}} (\nabla \psi) \quad (8.20)$$

and therefore

$$\nabla' \vec{\mathcal{T}} = \vec{\mathcal{T}} \nabla \quad (8.21)$$

then

$$\tilde{\psi}' C R (\nabla \psi)' = \tilde{\psi} C R (\nabla \psi) . \quad (8.22)$$

A natural choice for A is

$$A = \sum_{i=1}^3 (A(i) E(i) + \bar{A}(i) \bar{E}(i)) . \quad (8.23)$$

Then by (8.9) and (8.23) the invariant interaction (8.17) induces the following probability amplitude for the transition $\vec{n} \rightarrow \vec{n}'$:

$$\begin{aligned} & \langle n'_1 n'_2 n'_3 | \tilde{\psi} C R \mathcal{A} \psi | n_1 n_2 n_3 \rangle \\ &= \langle n'_1 n'_2 n'_3 | \sum_{i=1}^3 (\tilde{\psi}(i) H(i)) C R (\sum_{j=1}^3 \mathcal{A}(j) E(j)) (\sum_{k=1}^3 \psi(k) H(k)) | n_1 n_2 n_3 \rangle \\ & \quad + \text{contribution of } \bar{\mathcal{A}}(i) \bar{E}(i) \end{aligned} \quad (8.24)$$

$$\begin{aligned} &= \sum_{ijk} [\tilde{\psi}(i) C R \mathcal{A}(j) \psi(k)] \langle n'_1 n'_2 n'_3 | H(i) E(j) H(k) | n_1 n_2 n_3 \rangle \\ & \quad + \text{contribution of } \bar{\mathcal{A}}(i) \bar{E}(i) \\ &= \sum_{ijk} [\tilde{\psi}(i) C R \mathcal{A}(j) \psi(k)] q_1^{n_i + n_k} \alpha_i \alpha_k \langle n'_1 n'_2 n'_3 | E(j) | n_1 n_2 n_3 \rangle \\ & \quad + \text{contribution of } \bar{\mathcal{A}}(i) \bar{E}(i) \end{aligned} \quad (8.25)$$

The sum on (ijk) is restricted by (4.10) to the following combinations

$$\begin{aligned} j = 1 & \quad (i, k) = 1, 2, 3 \\ j = 2 & \quad (i, k) = 1, 2 \\ j = 3 & \quad (i, k) = 2, 3 \end{aligned} \quad (8.26)$$

It is also limited by the following selection rules

$$\begin{aligned} \langle \vec{n}' | \bar{E}(1) | \vec{n} \rangle &= 0 \quad \text{unless} \quad n'_1 = n_1 + 1 \quad n'_3 = n_3 + 1 \quad n'_2 = n_2 + 2 \\ \langle \vec{n}' | \bar{E}(2) | \vec{n} \rangle &= 0 \quad n'_1 = n_1 + 1 \quad n'_2 = n_2 + 1 \quad n'_3 = n_3 \\ \langle \vec{n}' | \bar{E}(3) | \vec{n} \rangle &= 0 \quad n'_1 = n_1 \quad n'_2 = n_2 + 1 \quad n'_3 = n_3 + 1 \end{aligned} \quad (8.27)$$

When $q = 1$ one has simply

$$\tilde{\psi} C \mathcal{A} \psi. \quad (8.28)$$

The new factors when $q \neq 1$ include

$$q_1^{n(i) + n(j)} \alpha_i \alpha_j = (m_i m_j)^{1/2} / M \quad (8.29)$$

as well as a similar factor from $\langle \vec{n}' | E | \vec{n} \rangle$, together implying that the associated probabilities for absorption and emission of a pair of heavy spinor particles by the vector particles proportional to the (product)² of the spinor masses.

We have described the elements of a formalism resembling the current non-Abelian gauge theories. The Lagrangian of this model is built out of the three H -fields and the three E -fields. If the H -fields are spinor fields, they are quarklike and if the three E -fields are gauge vectors, then they resemble gluons. If the interaction between the “quarks” and “gluons” is given by (8.15), then according to the selection rules (8.27) the “quarks” cannot be singly excited.

When the Lie group is deformed, the group (g) and the group algebra (u) undergo quite different deformations; and as a result the full deformed theory contains two sectors, one coming from g and the other from u . In this paper we have discussed the g -sector. The particles lying in this sector become invisible in the $q = 1$ limit and are nearly invisible if q is close to this limit. The u -sector, which we have not discussed here, results from the deformation of the algebra (u). In the $SU_q(2)$ case the deformed generators ($J_{(\pm)}^q, J_{(3)}^q$) satisfy

$$\begin{aligned} (J_3^q, J_{\pm}^q) &= \pm J_{\pm}^q \\ (J_+^q, J_-^q) &= \frac{1}{2}[2J_3^q]_q \\ [2J_3^q]_q &= \frac{q^{2J_3^q} - q^{-2J_3^q}}{q - q^{-1}} . \end{aligned} \tag{8.30}$$

As $q \rightarrow 1$ these relations approach the corresponding relations for the Lie algebra so that the usual Yang-Mills theory is recovered. The deviation of the u -sector from the standard theory is accessible by perturbation theory if q is near unity.

More interesting, however, if q is near unity, is the g sector which contains the hypothetical, exotic particles discussed in this paper and which are nearly invisible if q is near unity. We have not discussed the u -sector in this paper.

References.

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